A Manifold Minimization Principle for Physical Networks

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Physical networks [1], such as blood vessels, corals, plants, and neurons, are network-based physical objects that notably differ from abstract networks. Not only that physical networks typically have different traditional network metrics (such as distributions of degrees and cycles) than abstract networks [2], they are also endowed with rich geometry, such as distance, crossing angle, curvature, etc., that cannot be described by adjacency matrix only [1]. Despite an enormous body of studies on the *principle of formation* of abstract networks, e.g., the preferential attachment [3] or the fitness [4] mechanisms, there is not much literature on the principle of formation of physical networks, by which the nodes and links, rather than *a priori* defined, should naturally emerge from some geometric principle.

One such principle is wiring minimization, a.k.a. the Steiner problem [5], aiming at linking all destinations (terminals) with minimum total link length, which has long been considered as the physical interpretation of emerging network structures in complex systems such as neurons [6] and ant tunnels [7]. Promisingly, the solution of the Steiner problem naturally induces a tree-network structure, which nonetheless predicts the following geometric characteristics: (1) the tree only has degree ≤ 3 (i.e., bifurcation only); (2) all bifurcation angles are 120 degree (except at terminals); (3) all bifurcations are planar (even in higher-dimensional ambient space). By contrast, the Steiner predictions (1) and (2) are frequently violated in real physical networks. For example, nodes of degree 4 (trifurcation) cannot appear in the Steiner solution, since any degree-4 node can be locally replaced by two degree-3 nodes that produce a shorter overall link length, yet trifurcations are frequently observed in different biological systems [8]. Also, while most bifurcations are planar [9], the bifurcating branches are found usually not crossing at 120 degree, but many times even at 180 degree, a "sprouting" behavior that is frequently observed in, for example, blood vessels (sprouting angiogenesis) [10].

This prompts us to look into other candidates as the principle of formation of physical networks. Recent advance in physical networks indicates that treating links as having shapes leads to completely different geometric characteristics than as shapeless wires [1]. We are thus motivated to promote a graph to a higher-dimensional geometric object, namely, a smooth manifold [11]—a topological space that is everywhere locally similar enough to some Euclidean space, where we can define calculus and calculate geometric quantities. This leads to a principle of formation on $d \geq 2$ dimensions—a *manifold minimization principle* that, as we will see, gives rise to a tree-network solution that is analogous to the Steiner solution, yet need not follow the Steiner predictions (1) and (2), in full accordance with observations in real-world physical networks.

Manifold minimization principle.—Here, we focus on $d = 2$ manifolds, i.e., surfaces. *Surface area minimization* (Plateau's problem) has been extensively studied, which seems a natural generalization of the 1D Steiner problem if we can fix terminals as boundaries and study the minimal surface connecting them [Fig. 1(a)]. Unfortunately, the existence of long physical links is forbidden in Plateau's problem. Indeed, if we fix two parallel and identical circles as boundaries, then the minimal surface that connects the boundaries is a physically disconnected Goldschmidt solution [14] when $d/w > 0.168$ [Fig. 1(a)], where d is the distance between the two circles and w is the circumference of each circle. As a comparison, physical links in a real biological network typically have a much larger ratio $d/w \approx 10^0 \sim 10^1$. This indicates that real physical networks do not follow a surface minimization principle based merely on minimizing the area with no other constraints.

A key feature of physical networks is that the links must maintain *transportational functionality*: it is a necessary condition for physical links to have a physically connected skin in order to transport nutrients (e.g., tree bark) or signals (e.g., neuronal membrane). This prompts us to use the length scale w as a constraint and consider a systolic surface minimization problem: we require that every *systole* of the surface, defined as the shortest closed curve(s) on the surface that cannot be continuously contracted to a point because of topological holes it essentially surrounds, must have length w [Fig. 1(a)].

In the case of the circles as boundaries [Fig. 1(a)], the cylinder is a trivial solution to the systolic surface minimization problem. The surface maintains transporta-

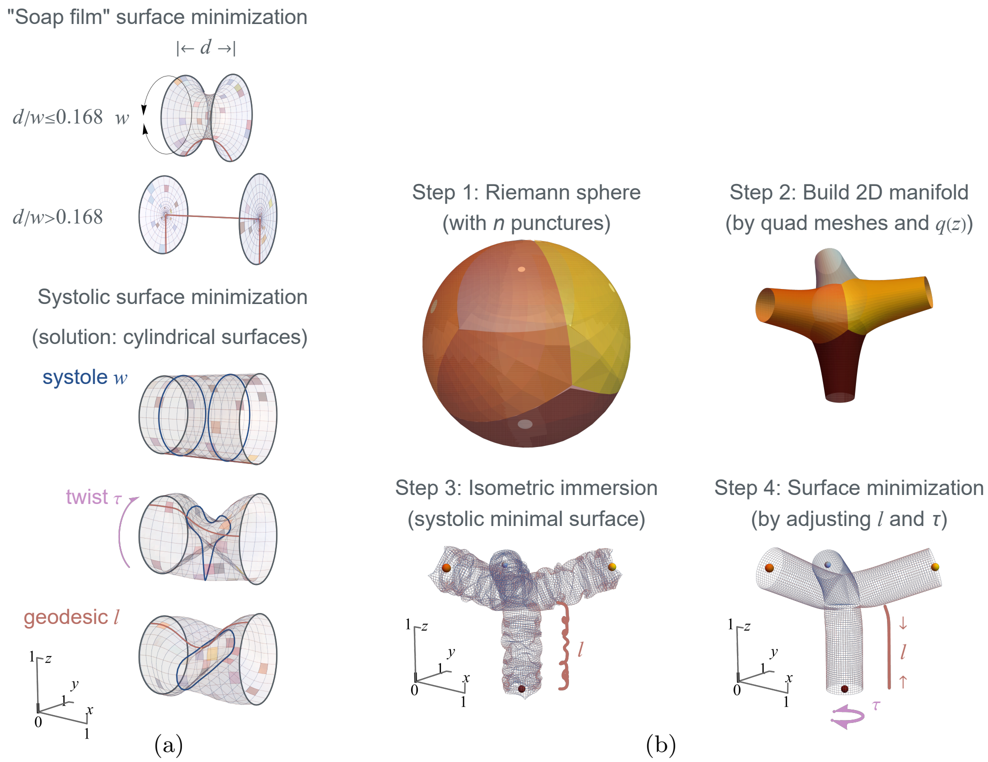

FIG. 1: **Manifold minimization.** **(a)** When $d/w > 0.168$, traditional "soap-film" minimal surface that connects the two circle boundaries degenerates from a catenoid to two planar disks and destroys transportational functionality. Instead, keeping the length of every systole (shortest closed trajectory) fixed as w, the systolic minimal surface maintains transportational functionality. In general, every cylindrical surface as well as their combinations (by sewing their ends to form a tree network) is a systolic minimal surface. **(b)** Systolic surface minimizer: (Step 1) Choose n punctures on the Riemann sphere, where n is the number of terminals. (Step 2) Find the corresponding Jenkins–Strebel quadratic differential $q(z)dz^2$ [12] and calculate the horizontal (blue) and vertical (red) trajectories. Along both trajectories square-like quad meshes [13] are tiled over the manifold, giving rise to multiple charts (different colors) that are cylindrical surfaces (by letting each quad mesh have the same size). (Step 3) Fix n terminals accordingly in the 3D Euclidean space. Immerse the manifold into the Euclidean space isometrically. (Step 4) While keeping the immersion isometric, adjust l and τ of each cylindrical surface so that the overall surface area is minimized.

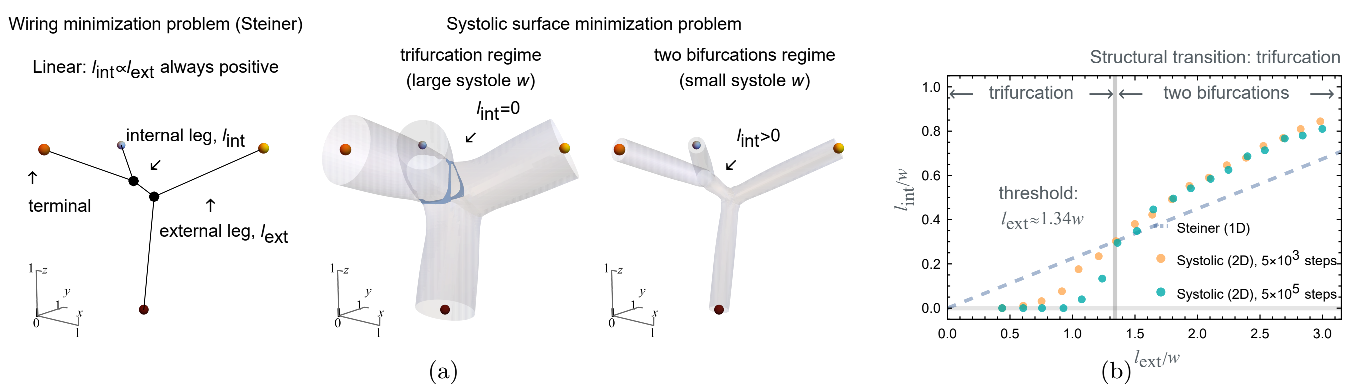

FIG. 2: **Trifurcation.** **(a)** Unlike Steiner graphs, systolic minimal surfaces allow trifurcation. **(b)** Given $n = 4$ terminals following perfect tetrahedral geometry, rather than linearly increasing with the internal leg length l_{int} (as in Steiner graph), the external leg length l_{ext} changes abruptly from the trifurcation regime ($l_{\text{int}} = 0$) to the double bifurcation regime ($l_{\text{int}} > 0$), crossing near the threshold of $l_{\text{int}} \approx 1.34w$.

tional functionality for finite w, its area always linearly scaling with its length. Although it is difficult to find a general numerical solution given general boundary conditions, it is proved that every *cylindrical surface*, which

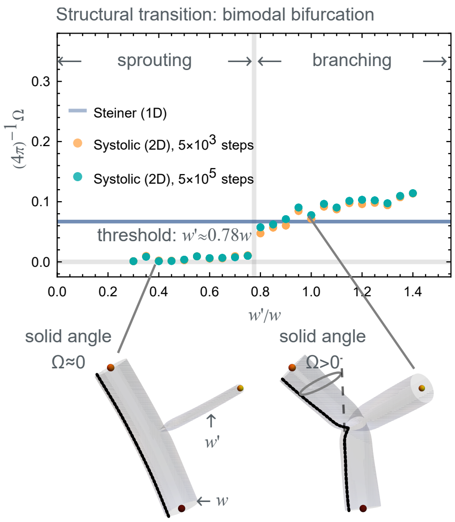

FIG. 3: **Bimodal bifurcation.** A bimodal bifurcation denotes a bifurcation with two typical but different systoles, one systole w for two of the external legs and another systole w' for the third external leg. Under manifold minimization, we observe a structural transition from the sprouting regime ($w'/w \lesssim 0.78$) to the branching regime ($w'/w \gtrsim 0.78$).

has a flat Riemannian metric everywhere (equivalent to a wrapped piece of flat paper but not necessarily a cylinder), is a *systolic minimal surface*, a special solution to our minimization problem [12]. Moreover, any 2D manifold, constructed from a tree graph of cylindrical surfaces as charts, is itself a systolic minimal surface too [15].

Hence, the systolic surface minimization problem reduces to a problem of first constructing a tree of cylindrical surfaces in the ambient space, i.e., an *isometric immersion* problem [13] that can be efficiently solved using the quad-mesh representation by letting each mesh not only be a square but also have the same size [13]. Given fixed boundaries, after finding all possible systolic minimal surfaces (i.e., with different twist τ or different l [Fig. 1(b)]) as solutions, the optimal solution is reached by selecting the systolic minimal surface with the smallest surface area from all possible combinations of l, τ, and node geometries.

Trifurcation.—Now we consider the systolic surface minimization problem with $N = 4$ terminals, located at the four corners of a perfect tetrahedron. Only l_{ext} and l_{int}, initially chosen to match the Steiner solution in the $w \to 0$ limit, are freely adjustable for surface minimization. When the systole $w > 0$, we find that l_{ext} and l_{int} follow a nonlinear relation that differs from Steiner's trivial linear relation (Fig. 2). When w is small (l_{ext}/w is large), the geodesic l_{int} of the internal leg remains positive and slightly above the Steiner prediction; when w is large (l_{ext}/w is small), however, l_{int} approaches zero. This indicates that a structural transition happens near a threshold of $l_{\text{ext}}/w \approx 1.34$, below which the degeneration of l_{int} signals the emergence of trifurcation.

Bimodal bifurcation.—A possible generalization of the systolic surface minimization is to constrain different physical links by different systoles. Here, we investigate the simplest case of a *bimodal* bifurcation that has different systoles, one systole w for two external legs and another w' for the third external leg, with all $N = 3$ terminals located at the three corners of an equilateral triangle (Fig. 3). Our algorithmic solution predicts not only that the branching angle (the solid angle Ω between the two same-systole external legs) correlates positively with the ratio of systoles w'/w, but also that a structural transition emerges near $w'/w \approx 0.78$, distinguishing two long-hypothesized different morphological regimes: [16] the "sprouting" regime (mode I [16]), where the branching angle is strictly zero; and the "branching" regime (mode II [16]), where the branching angle starts increasing with w'/w. Note also that bimodal bifurcations remain planar, hence do not violate the geometric constraint of planarity [9].

[1] N. Dehmamy, S. Milanlouei, and A.-L. Barabási, Nature **563**, 676 (2018).
[2] M. Barthélemy, Physics Reports **499**, 1 (2011).
[3] A.-L. Barabási and R. Albert, Science **286**, 509 (1999).
[4] G. Caldarelli, A. Capocci, P. De Los Rios, and M. A. Muñoz, Physical Review Letters **89**, 258702 (2002).
[5] F. K. Hwang, D. S. Richards, and P. Winter, *The Steiner Tree Problem*, 1st ed. (Elsevier, Amsterdam, The Netherlands, 1992).
[6] J. M. L. Budd, K. Kovács, A. S. Ferecskó, P. Buzás, U. T. Eysel, and Z. F. Kisvárday, PLOS Computational Biology **6**, e1000711 (2010).
[7] T. Latty, K. Ramsch, K. Ito, T. Nakagaki, D. J. T. Sumpter, M. Middendorf, and M. Beekman, Journal of The Royal Society Interface **8**, 1298 (2011).
[8] P. Bradley and M. Berry, Brain Research **109**, 133 (1976).
[9] Y. Kim, R. Sinclair, N. Chindapol, J. A. Kaandorp, and E. D. Schutter, PLOS Computational Biology **8**, e1002474 (2012).
[10] J. C. Chappell, D. M. Wiley, and V. L. Bautch, Seminars in Cell & Developmental Biology Mechanochemical Cell Biology, **22**, 1005 (2011).
[11] A. I. Bobenko, J. M. Sullivan, P. Schröder, and G. M. Ziegler, eds., *Discrete Differential Geometry*, 1st ed., Oberwolfach Seminars, Vol. 38 (Birkhäuser, Basel, Switzerland, 2008).
[12] K. Strebel, *Quadratic Differentials*, 1st ed. (Springer, Heidelberg, Germany, 1984).
[13] C. Jiang, C. Wang, F. Rist, J. Wallner, and H. Pottmann, ACM Transactions on Graphics **39**, 128:128:1 (2020).
[14] J. C. C. Nitsche, Journal of Mathematics and Mechanics **13**, 659 (1964).
[15] B. Zwiebach, Physics Letters B **241**, 343 (1990).
[16] M. Zamir, Journal of Theoretical Biology **62**, 227 (1976).